Separované diferenciálne rovnice

- je u nich charakteristické oddelenie funkcie od ***x*** od funkcie od ***y***:

$$P(x) + Q(y) \cdot y' = 0$$

$$\int P(x)dx + \int Q(y)dy = c \qquad c \text{ je konštanta.}$$

Príklad č.1:

$$\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} \cdot y' = 0$$

$$\int \frac{x}{\sqrt{1+x^2}}dx + \int \frac{y}{\sqrt{1+y^2}}dy = c \quad \Bigg/ \begin{array}{l} \sqrt{1+x^2} = t \\ 1+x^2 = t^2 \\ 2xdx = 2tdt \end{array} \Bigg/ \begin{array}{l} \sqrt{1+y^2} = u \\ 1+y^2 = u^2 \\ 2ydy = 2udu \end{array}$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx + \frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}dy = c$$

$$\frac{1}{2}\int \frac{t}{t}dt + \frac{1}{2}\int \frac{u}{u}du = c$$

$$\int dt + \int du = 2c$$

$$t + u = 2c$$

$$\underline{\underline{\sqrt{1+x^2} + \sqrt{1+y^2} = 2c}}$$

Príklad č.2:

$$2yy' - 4x^3 = 0$$

$$\int -4x^3dx + \int 2ydy = c$$

$$-4\frac{1}{4}x^4 + 2\frac{1}{2}y^2 = c$$

$$y^2 - x^4 = c$$

$$\underline{\underline{y = \sqrt{c - x^4}}}$$

Príklad č.3:

$$\frac{1}{x^2}\cos\frac{1}{x} + y' = 0 \Big/ y\left(\frac{2}{\pi}\right) = 3 \Big/ x \neq 0$$

$$\int \frac{1}{x^2}\cos\frac{1}{x}dx + \int dy = k \Big/ t = \frac{1}{x}$$

$$-\int \cos t dt + y = k \Rightarrow dt = -\frac{1}{x^2}dx$$

$$y = c + \sin\frac{1}{x}$$

$$3 = c + \sin\frac{\pi}{2}$$

$$3 = c + 1 \Rightarrow c = 2$$

$$\underline{\underline{y = 2 + \sin\frac{1}{x} \qquad x \neq 0}}$$

Separovateľné diferenciálne rovnice

- charakteristické zmiešanými funkciami od ***x*** aj od ***y***. Po jednoduchej úprave dostávame separovanú diferenciálnu rovnicu:

$$P_1(x)\cdot P_2(y)+Q_1(x)\cdot Q_2(y)\cdot y'=0$$
$$\frac{P_1(x)}{Q_1(x)}+\frac{Q_2(y)}{P_2(y)}\cdot y'=0$$
$$\int\frac{P_1(x)}{Q_1(x)}dx+\int\frac{Q_2(y)}{P_2(y)}dy=c\,;\quad c \text{ je konštanta}$$

Príklad č.4:

$$y+\left(1-x^2\right)\cdot y'=0$$
$$\frac{1}{1-x^2}+\frac{1}{y}\cdot y'=0$$
$$\int\frac{1}{1-x^2}dx+\int\frac{1}{y}dy=c$$
$$\int\left(\frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x}\right)dx+\int\frac{1}{y}dy=c$$
$$-\frac{1}{2}\int\frac{-1}{1-x}dx+\frac{1}{2}\int\frac{1}{1+x}dx+\int\frac{1}{y}dy=c$$
$$-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|+\ln|y|=c$$
$$\ln|1+x|^{\frac{1}{2}}-\ln|1-x|^{\frac{1}{2}}+\ln|y|=c\Big/c=\ln c^*$$
$$\ln\sqrt{\frac{1+x}{1-x}}+\ln|y|=\ln c^*$$
$$\ln|y|=\ln c^*-\ln\sqrt{\frac{1+x}{1-x}}$$
$$\ln|y|=\ln c^*+\ln\sqrt{\frac{1-x}{1+x}}$$
$$\underline{\underline{y=c^*\cdot\sqrt{\frac{1-x}{1+x}}}}$$

Príklad č.5:

$x \cdot y' - y = 0$ pre $x = -2$, $y = 4$

$\frac{1}{x} - \frac{1}{y} y' = 0$

$\int \frac{1}{x} dx - \int \frac{1}{y} dy = -\ln c$

$\ln x - \ln y = -\ln c$

$\ln y = \ln x + \ln c$

$\underline{y = cx}$

$4 = c \cdot (-2)$

$c = -2$

$\underline{\underline{y = -2x}}$

Príklad č.6:

$xy' + y = 0$; pre $x = -2$; $y = 4$

$\frac{1}{x} + \frac{1}{y} y' = 0$

$\int \frac{1}{x} dx + \int \frac{1}{y} dy = \ln c$

$\ln x + \ln y = \ln c$

$\ln y = \ln c - \ln x$

$\underline{y = \frac{c}{x}}$

$4 = \frac{c}{-2}$

$c = -8$

$\underline{\underline{y = \frac{-8}{x}}}$

Príklad č.7:

$$x \cdot (1-y) = (1+x) \cdot y'$$

$$x \cdot (1-y) - (1+x) \cdot y' = 0$$

$$\frac{x}{x+1} + \frac{-1}{1-y} y' = 0$$

$$\int \frac{x}{x+1} dx + \int \frac{-1}{1-y} dy = c \quad \Big/ \begin{matrix} 1+x=t \quad \Rightarrow \quad x = t-1 \\ dx = dt \end{matrix}$$

$$\int \frac{t-1}{t} dx + \int \frac{-1}{1-y} dy = c$$

$$\int 1 - \frac{1}{t} dx + \int \frac{-1}{1-y} dy = c$$

$$t - \ln|t| + \ln|1-y| = c$$

$$1 + x - \ln|1+x| + \ln|1-y| = c \qquad \Big/ \quad c - 1 = c^*$$

$$x - \ln|1+x| + \ln|1-y| = c^*$$

$$\ln|1-y| = c^* - x + \ln|1+x| \qquad \Big/ \quad x = \ln e^x$$

$$\ln|1-y| = c^* - \ln e^x + \ln|1+x| \qquad \Big/ \cdot (-1) \qquad \Big/ c^* = \ln c_1$$

$$-\ln|1-y| = -\ln c_1 + \ln e^x - \ln|1+x|$$

$$-\ln|1-y| = \ln \frac{e^x}{c_1 \cdot (1+x)}$$

$$\ln \frac{1}{1-y} = \ln \frac{e^x}{c_1 \cdot (1+x)}$$

$$\frac{1}{1-y} = \frac{e^x}{c_1 \cdot (1+x)}$$

$$\underline{\underline{y = 1 - \frac{c_1 \cdot (1+x)}{e^x}}}$$

Príklad č.8:

$$x^2 y' + y = 0$$

$$\frac{1}{x^2} + \frac{1}{y} y' = 0$$

$$\int \frac{1}{x^2} dx + \int \frac{1}{y} dy = \ln c$$

$$\frac{-1}{x} + \ln y = \ln c$$

$$\ln y = \ln c + \frac{1}{x}$$

$$\ln y = \ln c + \ln e^{\frac{1}{x}}$$

$$\underline{\underline{y = c \cdot \sqrt[x]{e}}}$$

Príklad č.9:

$$2y'\sqrt{x} = y$$

$$2y'\sqrt{x} - y = 0$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{y} y' = 0$$

$$\frac{1}{2}\int x^{-\frac{1}{2}} dx - \int \frac{1}{y} dy = -\ln c$$

$$\sqrt{x} - \ln y = -\ln c$$

$$\sqrt{x} + \ln c = \ln y$$

$$\ln e^{\sqrt{x}} + \ln c = \ln y$$

$$\underline{\underline{y = c \cdot e^{\sqrt{x}}}}$$

Príklad č.10:

$$x^3 y' - 2y = 0$$

$$\frac{1}{y} y' - \frac{2}{x^3} = 0$$

$$-\int \frac{2}{x^3} dx + \int \frac{1}{y} dy = c$$

$$x^{-2} + \ln y = c$$

$$y = e^{c - \frac{1}{x^2}} \Big/ \quad c = \ln k \quad \Rightarrow \quad \underline{\underline{y = k \cdot \frac{1}{\sqrt[x^2]{e}}}}$$

Homogénne diferenciálne rovnice

- zavádzame novú premennú ***u***.

$$y' = f(x)$$
$$y' = F\left(\frac{y}{x}\right)$$
$$u = \frac{y}{x} \quad \Rightarrow \quad y = ux$$
$$y' = u'x + u$$
$$F'(u) = u'x + u$$

Príklad č.11:

$$x - y + xy' = 0 \quad \Big/ \cdot \frac{1}{x}$$
$$1 - \frac{y}{x} + y' = 0 \quad \Big/ u = \frac{y}{x}; \quad y' = u'x + u$$
$$1 - u + u + u'x = 0$$

$1 + u'x = 0$ čo je separovateľná diferenciálna rovnica

$\frac{1}{x} + u' = 0$ čo je separovaná diferenciálna rovnica

$$\int \frac{1}{x} dx + \int du = c$$
$$\ln|x| + u = c$$
$$\ln|x| + \frac{y}{x} = c$$
$$\underline{\underline{y = x \cdot (c - \ln|x|)}}$$

Príklad č.12:

$$y^2 - xy + \left(x^2 + xy\right)y' = 0 \qquad \Big/ \cdot \frac{1}{x^2}$$

$$\frac{y^2}{x^2} - \frac{y}{x} + \frac{x^2 + xy}{x^2}y' = 0$$

$$\frac{y^2}{x^2} - \frac{y}{x} + \left(1 + \frac{y}{x}\right)y' = 0 \qquad \Big/ u = \frac{y}{x}; \quad y' = u + u'x$$

$$u^2 - u + (1 + u)\cdot(u + u'x) = 0$$

$$u^2 - u + u + u'x + u^2 + u'ux = 0$$

$$2u^2 + (1 + u)xu' = 0$$

$$\frac{2}{x} + \frac{1 + u}{u^2}u' = 0$$

$$\int \frac{2}{x}dx + \int \frac{1 + u}{u^2}du = c$$

$$2\ln|x| + \ln|u| - \frac{1}{u} = c$$

$$2\ln|x| + \ln\left|\frac{y}{x}\right| - \frac{x}{y} = c$$

$$2\ln|x| + \ln|y| - \ln|x| - \frac{x}{y} = c$$

$$\ln|x| + \ln|y| - \frac{x}{y} = c$$

$$\ln|xy| = c + \frac{x}{y}$$

$$\ln xy = \ln e^{c + \frac{x}{y}}$$

$$\underline{\underline{y = \frac{e^{c + \frac{x}{y}}}{x}}}$$

Príklad č.13:

$$(x-y)y'-y=0 \qquad \Big/ \cdot \frac{1}{x}$$

$$\left(1-\frac{y}{x}\right)y'-\frac{y}{x}=0 \qquad \Big/ \cdot u=\frac{y}{x}; \qquad y'=u+u'x$$

$$(1-u)\cdot(u+u'x)-u=0$$

$$u'x+u-u'ux-u^2-u=0$$

$$u'x\cdot(1-u)-u^2=0$$

$$-\frac{1}{x}+\frac{1-u}{u^2}u'=0$$

$$-\int\frac{1}{x}dx+\int\frac{1-u}{u^2}du=-c$$

$$-\ln|x|-\frac{1}{u}-\ln|u|=-c$$

$$-\ln|x|-\frac{x}{y}-\ln\left|\frac{y}{x}\right|=-c$$

$$-\ln|x|-\frac{x}{y}-(\ln|y|-\ln|x|)=-c$$

$$\frac{x}{y}-\ln|y|=-c$$

$$c+\frac{x}{y}=\ln|y|$$

$$\underline{\underline{e^{c+\frac{x}{y}}=y}}$$

Príklad č.14:

$$x^2 + y^2 - 2xyy' = 0 \qquad \Big/ \frac{1}{x^2}$$

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} - \frac{2xy}{x^2} y' = 0 \qquad \Big/ u = \frac{y}{x} \; ; \quad y' = u + xu'$$

$$1 + u^2 - 2u(u + xu') = 0$$

$$1 + u^2 - 2u^2 - 2xuu' = 0$$

$$1 - u^2 - 2xuu' = 0 \qquad \Big/ \frac{1}{1-u^2}$$

$$1 - \frac{2ux}{1-u^2} u' = 0 \qquad \Big/ \frac{1}{x}$$

$$\frac{1}{x} - \frac{2u}{1-u^2} u' = 0$$

$$\int \frac{1}{x} dx + \int \frac{-2u}{1-u^2} du = \ln c$$

$$\ln|x| + \ln\left|1-u^2\right| = \ln c$$

$$\ln\left|1-u^2\right| = \ln c - \ln|x|$$

$$\ln\left|1-u^2\right| = \ln \frac{c}{x}$$

$$1 - u^2 = \frac{c}{x}$$

$$1 - \frac{y^2}{x^2} = \frac{c}{x}$$

$$1 - \frac{c}{x} = \frac{y^2}{x^2}$$

$$x^2 - cx = y^2$$

$$\underline{\underline{y = \pm\sqrt{x^2 - cx}}}$$

Príklad č.15:

$$2xy-\left(x^2-y^2\right)y'=0 \qquad \Big/ \frac{1}{x^2}$$

$$\frac{2y}{x}-\left(1-\frac{y^2}{x^2}\right)y'=0 \qquad \Big/ u=\frac{y}{x} \;; \quad y'=u+xu'$$

$$2u-\left(1-u^2\right)\left(u+xu'\right)=0$$

$$2u-u-xu'+u^3+xu^2u'=0$$

$$u+u^3+xu'\left(u^2-1\right)=0 \qquad \Big/ \frac{1}{u+u^3}$$

$$1+xu'\frac{u^2-1}{u+u^3}=0$$

$$\frac{1}{x}+\frac{u^2-1}{u+u^3}u'=0$$

$$\int\frac{1}{x}dx+\int\frac{u^2-1}{u+u^3}du=\ln c$$

$$\ln|x|+\int\left(-\frac{1}{u}+\frac{2u}{1+u^2}\right)du=\ln c$$

$$\ln|x|-\ln|u|+\ln\left|1+u^2\right|=\ln c$$

$$\ln|x|-\ln\left|\frac{y}{x}\right|+\ln\left|1+\frac{y^2}{x^2}\right|=\ln c$$

$$\ln\left|\frac{1+\frac{y^2}{x^2}}{\frac{y}{x}}\right|=\ln\frac{c}{x}$$

$$\frac{1+\frac{y^2}{x^2}}{\frac{y}{x}}=\frac{c}{x}$$

$$\frac{\left(x^2+y^2\right)x}{x^2y}=\frac{c}{x}$$

$$\underline{\underline{\frac{x^2+y^2}{y}=c}}$$

Príklad č.16:

$$xy+\left(y^2-xy+x^2\right)y'=0 \qquad \Big/ \frac{1}{x^2}$$

$$\frac{y}{x}+\left(\frac{y^2}{x^2}-\frac{y}{x}+1\right)y'=0 \qquad \Big/ u=\frac{y}{x}$$

$$u+\left(u^2-u+1\right)\cdot\left(u+xu'\right)=0$$

$$u+u^3+xu^2u'-u^2-xuu'+u+xu'=0$$

$$2u+u^3-u^2+xu'\left(u^2-u+1\right)=0 \qquad \Big/ \frac{1}{x} \quad \Big/ \frac{1}{u^3-u^2+2u}$$

$$\frac{1}{x}+\frac{u^2-u+1}{u^3-u^2+2u}u'=0$$

$$\int\frac{1}{x}dx+\int\frac{u^2-u+1}{u^3-u^2+2u}du=c$$

Integrál podľa *du* budeme riešiť pomocou rozkladu na parciálne zlomky:

$$\frac{A}{u}+\frac{Bu+C}{u^2-u+2}=\frac{Au^2-Au+2A+Bu^2+Cu}{u\cdot\left(u^2-u+2\right)}=\frac{u^2-u+1}{u\cdot\left(u^2-u+2\right)}$$

$$Au^2+Bu^2=u^2 \Rightarrow \frac{1}{2}+B=1\Rightarrow B=\frac{1}{2}$$

$$-Au+Cu=-u\Rightarrow -\frac{1}{2}+C=-1\Rightarrow C=-\frac{1}{2}$$

$$2A=1\Rightarrow A=\frac{1}{2}$$

Pokračujeme v riešení integrálov:

$$\ln|x|+\frac{1}{2}\int\left(\frac{1}{u}+\frac{u-1}{u^2-u+2}\right)du=\ln c$$

$$\ln|x|+\frac{1}{2}\ln|u|+\frac{4}{4}\int\left(\frac{1}{2}\cdot\frac{u-1}{u^2-u+2}\right)du=\ln c$$

$$\ln|x|+\frac{1}{2}\ln|u|+\frac{1}{4}\ln\left|u^2-u+2\right|-\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2u-1}{\sqrt{7}}=\ln c$$

$$\ln|x|+\frac{1}{2}\ln\left|\frac{y}{x}\right|+\frac{1}{4}\ln\left|\frac{y^2}{x^2}-\frac{y}{x}+2\right|-\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2\frac{y}{x}-1}{\sqrt{7}}=\ln c$$

$$\ln|x|-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|y|+\frac{1}{4}\ln\left|y^2-xy+2x^2\right|-\frac{1}{4}\ln\left|x^2\right|=\ln c+\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2y-x}{\sqrt{7}x}$$

$$\ln\left|\sqrt{y}\right|+\ln\left|\sqrt[4]{y^2-xy+2x^2}\right|=\ln c+\ln e^{\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2y-x}{\sqrt{7}x}}$$

$$\underline{\underline{\sqrt{y}\cdot\sqrt[4]{y^2-xy+2x^2}=c\cdot\exp\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2y-x}{\sqrt{7}x}}}$$

Príklad č.17:

$$(x-y)y' = x+y$$

$$(x-y)y' - x - y = 0 \qquad \Big/ \frac{1}{x}$$

$$\left(1-\frac{y}{x}\right)y' - 1 - \frac{y}{x} = 0 \qquad \Big/ u = \frac{y}{x} \ ; \quad y' = u + xu'$$

$$(1-u)\cdot(u+xu') - 1 - u = 0$$

$$u + xu' - u^2 - xuu' - 1 - u = 0$$

$$-1 - u^2 - xu'(u-1) = 0$$

$$1 + u^2 + xu'(u-1) = 0 \qquad \Big/ \cdot\frac{1}{x}\cdot\frac{1}{1+u^2}$$

$$\frac{1}{x} + \frac{u-1}{1+u^2}u' = 0$$

$$\int \frac{1}{x}dx + \int\left(\frac{u}{1+u^2} - \frac{1}{1+u^2}\right)du = \ln c$$

$$\ln|x| + \frac{1}{2}\ln\left|1+u^2\right| - \operatorname{arctg} u = \ln c$$

$$\ln|x| + \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right| - \operatorname{arctg}\frac{y}{x} = \ln c$$

$$\ln|x| + \frac{1}{2}\ln\left|x^2+y^2\right| - \ln|x| - \operatorname{arctg}\frac{y}{x} = \ln c$$

$$\ln\sqrt{x^2+y^2} = \ln c + \operatorname{arctg}\frac{y}{x}$$

$$\ln\sqrt{x^2+y^2} = \ln c + \ln e^{\operatorname{arctg}\frac{y}{x}}$$

$$\underline{\underline{\sqrt{x^2+y^2} = c\cdot e^{\operatorname{arctg}\frac{y}{x}}}}$$

Lineárne diferenciálne rovnice

$y' + p(x)y = 0 \quad \Big/ \cdot \frac{1}{y}$

$\frac{y'}{y} + p(x) = 0$

$\ln|y| = -\int p(x)dx + \ln c$

$y = c \cdot e^{-\int p(x)dx}$

$y = \alpha \cdot e^{-\int p(x)dx} \qquad \alpha \in (-\infty, \infty)$

$\alpha = \alpha(x)$

S pravou stranou: $y' + p(x)y = g(x)$

Najprv riešime homogénnu rovnicu (bez pravej strany), potom:

$y' = \alpha' \cdot e^{-\int p(x)dx} + \alpha \cdot e^{-\int p(x)dx} \cdot (-p(x))$

dosaníme do pôvodnej rovnice za y' aj za y:

$\alpha' \cdot e^{-\int p(x)dx} + \alpha \cdot e^{-\int p(x)dx} \cdot (-p(x)) + p(x) \cdot \alpha \cdot e^{-\int p(x)dx} = g(x)$

$\alpha' \cdot e^{-\int p(x)dx} = g(x)$

$\alpha' = \frac{g(x)}{e^{-\int p(x)dx}} = g(x) \cdot e^{\int p(x)dx}$

$\alpha = \int g(x) \cdot e^{\int p(x)dx} dx$

$$y = \left[\int g(x) \cdot e^{\int p(x)dx} dx + k\right] \cdot e^{-\int p(x)dx}$$

Príklad č.18:

$y' + 2y = e^{3x}$ Najskôr budeme riešiť homogénnu rovnicu, t.j. bez pravej strany:

$$y' + 2y = 0$$

$$2 + \frac{1}{y}y' = 0$$

$$\int 2dx + \int \frac{1}{y}dy = \ln c$$

$$2x + \ln y = \ln c$$

$$\ln y = \ln e^{-2x} + \ln c$$

$$y = c \cdot e^{-2x}$$

$\underline{y = \alpha(x) \cdot e^{-2x}}$, čo je riešenie homogénnej diferenciálnej rovnice. Pokračujeme v riešení danej rovnice s pravou stranou:

$\underline{y' = \alpha'(x) \cdot e^{-2x} - 2e^{-2x}\alpha(x)}$ Dosadíme do pôvodnej rovnice za y' aj za y:

$$\alpha'(x) \cdot e^{-2x} - 2e^{-2x}\alpha(x) + 2e^{-2x}\alpha(x) = e^{3x}$$

$$\alpha'(x) = e^{5x}$$

$$\alpha(x) = \int e^{5x}dx$$

$\underline{\alpha(x) = \frac{1}{5}e^{5x} + k}$ kde k je integračná konštanta. Tento výsledok dosadíme do riešenia homogénnej lineárnej diferenciálnej rovnice a to je naše hľadané riešenie:

$$\underline{\underline{y = \left[\frac{1}{5}e^{5x} + k\right] \cdot e^{-2x}}}$$

Príklad č.19:

$$xy' - y = x^2 \quad \Big/ \cdot \frac{1}{x}$$

$$y' - \frac{y}{x} = x$$

$$y' - \frac{y}{x} = 0 \quad \Big/ \cdot \frac{1}{y}$$

$$\int \frac{1}{y} dy - \int \frac{1}{x} dx = \ln c$$

$$\ln y - \ln x = \ln c$$

$$y = cx$$

$$y = \alpha(x)x$$

$y' = \alpha'(x)x + \alpha(x)$ Posledné dve rovnice dosadím do upravenej pôvodnej diferenciálnej rovnice:

$$\alpha'(x)x + \alpha(x) - \frac{1}{x}\alpha(x)x = x$$

$$\alpha'(x) = 1$$

$$\alpha(x) = \int 1 dx = x + k$$

$$\underline{\underline{y = [x + k] \cdot x}}$$

Príklad č.20:

$$xy' - 2y = x^3 e^x \quad \Big/ \cdot \frac{1}{x} \cdot \frac{1}{y}$$

$$y' - \frac{2}{x} y = x^2 e^x$$

$$\int \frac{1}{y} dy - \int \frac{2}{x} dx = \ln c$$

$$\ln y - 2\ln x = \ln c$$

$$y = cx^2$$

$$y = \alpha(x)x^2$$

$y' = \alpha'(x)x^2 + 2\alpha(x)x$ Posledné dve rovnice dosadím do upravenej pôvodnej diferenciálnej rovnice:

$$\alpha'(x)x^2 + 2\alpha(x)x - \frac{2}{x}\alpha(x)x^2 = x^2 e^x$$

$$\alpha'(x) = e^x$$

$$\alpha(x) = \int e^x dx = e^x + k$$

$$\underline{\underline{y = [e^x + k] \cdot x^2}}$$

Príklad č.21:

$$xy' + y = x + 1 \quad \Big/ \cdot \frac{1}{x}$$

$$y' + \frac{y}{x} = \frac{x+1}{x}$$

$$y' + \frac{y}{x} = 0 \quad \Big/ \cdot \frac{1}{y}$$

$$\frac{1}{y} y' = -\frac{1}{x}$$

$$\int \frac{1}{y} dy = -\int \frac{1}{x} dx + \ln c$$

$$\ln y = -\ln x + \ln c$$

$$y = \frac{c}{x}$$

$$y = \frac{\alpha(x)}{x}$$

$$y' = \frac{\alpha'(x)x - \alpha(x)}{x^2}$$

Posledné dve rovnice dosadím do upravenej pôvodnej diferenciálnej rovnice:

$$\frac{\alpha'(x)x - \alpha(x)}{x^2} + \frac{1}{x}\frac{\alpha(x)}{x} = \frac{x+1}{x}$$

$$\alpha'(x) = x + 1$$

$$\alpha(x) = \int (x+1)dx = \frac{1}{2}x^2 + x + k$$

$$y = \left[\frac{1}{2}x^2 + x + k\right] \cdot \frac{1}{x}$$

Príklad č.22:

$$\left(x^2+1\right)y'+4xy=1$$

$$\frac{1}{y}y'+\frac{4x}{x^2+1}=\frac{1}{x^2+1}$$

$$\frac{1}{y}y'+\frac{4x}{x^2+1}=0$$

$$\int\frac{1}{y}dy=-\int\frac{4x}{x^2+1}dx+\ln c$$

$$\ln y=-2\ln\left|x^2+1\right|+\ln c$$

$$y=\frac{c}{\left(x^2+1\right)^2}$$

$$y=\frac{\alpha(x)}{\left(x^2+1\right)^2}$$

$$y'=\frac{\alpha'(x)\left(x^2+1\right)^2-2\alpha(x)\left(x^2+1\right)}{\left(x^2+1\right)^4}=\frac{\alpha'\left(x^2+1\right)-4x\alpha}{\left(x^2+1\right)^3}$$

Posledné dve rovnice dosadím do upravenej pôvodnej diferenciálnej rovnice:

$$\frac{\alpha'\left(x^2+1\right)-4x\alpha}{\left(x^2+1\right)^3}+\frac{4x}{x^2+1}\frac{\alpha}{\left(x^2+1\right)^2}=\frac{1}{x^2+1}$$

$$\alpha'=x^2+1$$

$$\alpha=\int\left(x^2+1\right)dx=\frac{1}{3}x^3+x+k$$

$$y=\left[\frac{1}{3}x^3+x+k\right]\cdot\frac{1}{\left(x^2+1\right)^2}$$

Príklad č.23:

$$(2x+1)y' + y = x$$

$$y' + \frac{y}{2x+1} = \frac{x}{2x+1}$$

$$\frac{1}{y}y' + \frac{1}{2x+1} = 0$$

$$\ln y + \frac{1}{2}\ln|2x+1| = \ln c$$

$$\ln y = -\ln\sqrt{2x+1} + \ln c$$

$$y = \frac{c}{\sqrt{2x+1}}$$

$$y = \frac{\alpha(x)}{\sqrt{2x+1}}$$

$$y' = \frac{\alpha'(x)\sqrt{2x+1} - \alpha(x)\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2x+1}}\cdot 2}{2x+1} = \frac{\alpha'\sqrt{2x+1} - \frac{\alpha}{\sqrt{2x+1}}}{2x+1}$$

Posledné dve rovnice dosadíme do upravenej pôvodnej diferenciálnej rovnice:

$$\frac{\alpha'\sqrt{2x+1}}{2x+1} - \frac{\alpha}{\left(\sqrt{2x+1}\right)^3} + \frac{\alpha}{\left(\sqrt{2x+1}\right)^3} = \frac{x}{2x+1}$$

$$\alpha' = \frac{x}{\sqrt{2x+1}}$$

$$\alpha = \int\frac{x}{\sqrt{2x+1}}dx \qquad \Big/ \text{substitúcia}: 2x+1 = t \Rightarrow 2dx = dt \Rightarrow x = \frac{t-1}{2}$$

$$\alpha = \int\frac{\frac{t-1}{2}}{\sqrt{t}}\frac{dt}{2} = \frac{1}{2}\int\left(\frac{t}{2t^{\frac{1}{2}}} - \frac{1}{2t^{\frac{1}{2}}}\right)dt = \frac{1}{4}\int\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)dt$$

$$\alpha = \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + k$$

$$\alpha = \frac{1}{6}\sqrt{t^3} - \frac{1}{2}\sqrt{t} + k$$

$$\alpha = \frac{1}{6}\sqrt{(2x+1)^3} - \frac{1}{2}\sqrt{2x+1} + k = \frac{1}{2}\sqrt{2x+1}\left(\frac{1}{3}(2x+1) - 1\right) + k$$

$$y = \left[\frac{1}{2}\sqrt{2x+1}\left(\frac{1}{3}(2x+1) - 1\right) + k\right]\cdot\frac{1}{\sqrt{2x+1}}$$

$$\underline{\underline{y = \frac{x-1}{3} + \frac{k}{\sqrt{2x+1}}}}$$

Príklad č.24:

$y' + \frac{2x}{1+x^2} y = \frac{2x^2}{1+x^2}$ Riešime homogénnu diferenciálnu rovnicu, t.j. bez pravej strany.

$$\frac{y'}{y} = -\frac{2x}{1+x^2}$$

$$\ln y = \ln c - \ln\left|1+x^2\right|$$

$$y = \frac{c}{1+x^2}$$

$$y = \frac{\alpha(x)}{1+x^2}$$

$$y' = \frac{(1+x^2)\alpha'(x) - 2x\alpha(x)}{(1+x^2)^2}$$

$$\frac{(1+x^2)\alpha'(x) - 2x\alpha(x)}{(1+x^2)^2} + \frac{2x}{1+x^2}\frac{\alpha(x)}{1+x^2} = \frac{2x^2}{1+x^2}$$

$$\frac{\alpha'(x)}{1+x^2} = \frac{2x^2}{1+x^2}$$

$$\alpha'(x) = 2x^2$$

$$\alpha(x) = \int 2x^2 dx$$

$$\alpha(x) = \frac{2}{3}x^3 + k$$

$$y = \frac{\frac{2}{3}x^3 + k}{1+x^2}$$

$$\underline{\underline{y = \frac{2x^3}{3(1+x^2)} + \frac{k}{1+x^2}}}$$

Bernouliho diferenciálne rovnice

$$y' + p(x)y = g(x)y^{\alpha} \qquad \Big/ \cdot y^{-\alpha}$$
$$y' \cdot y^{-\alpha} + p(x)y^{1-\alpha} = g(x)$$
$$z = y^{1-\alpha}$$
$$z' = (1-\alpha)y^{-\alpha}y'$$
$$\frac{z'}{1-\alpha} = y^{-\alpha}y'$$
$$\frac{z'}{1-\alpha} + p(x)z = g(x)$$ čo je lineárna diferenciálna rovnica.

Príklad č.25:

$$xy' - y = x^2y^{-1} \qquad \Big/ \cdot y$$
$$xyy' - y^2 = x^2$$
$$z = y^{1-(-1)} = y^2$$
$$z' = 2yy' \Rightarrow y' = \frac{z'}{2y}$$
$$xy\frac{z'}{2y} - z = x^2$$
$$\frac{x}{2}z' - z = x^2$$
$$z' - \frac{2z}{x} = 2x \qquad (1)$$
$$\frac{z'}{z} - \frac{2}{x} = 0$$
$$\ln z = 2\ln x + \ln c$$
$$z = cx^2$$
$$z = \alpha(x) \cdot x^2$$
$$z' = \alpha'(x) \cdot x^2 + 2x\alpha(x)$$ Posledné dve rovnice dosadíme do rovnice (1).
$$\alpha'(x) \cdot x^2 + 2x\alpha(x) - \frac{2\alpha(x) \cdot x^2}{x} = 2x$$
$$\alpha'(x) = \frac{2}{x}$$
$$\alpha = 2\ln x + k = \ln x^2 + k$$
$$z = (\ln x^2 + k)x^2 = y^2$$
$$\underline{\underline{y = x\sqrt{\ln x^2 + k}}}$$

Príklad č.26:

$$y' + \frac{y}{1+x} = -y^2 \quad \Big/ \cdot y^{-2}$$

$$y'y^{-2} + \frac{1}{y(1+x)} = -1$$

$$z = y^{1-2} = y^{-1}$$

$$z' = -y^{-2}y' \Rightarrow y' = -y^2 z'$$

$$-y^2 y^{-2} z' + \frac{z}{1+x} = -1$$

$$z' - \frac{z}{1+x} = 1 \qquad (1)$$

Riešime homogénnu diferenciálnu rovnicu, t.j. bez pravej strany:

$$z' - \frac{z}{1+x} = 0$$

$$\frac{z'}{z} - \frac{1}{1+x} = 0$$

$$\ln z = \ln c + \ln|1+x|$$

$$z = \alpha|1+x|$$

$$z' = \alpha'(1+x) + \alpha$$

Posledné dve rovnice dosadíme do rovnice (1):

$$\alpha'(1+x) + \alpha - \frac{\alpha(1+x)}{1+x} = 1$$

$$\alpha' = \frac{1}{1+x}$$

$$\alpha = \ln|1+x| + k$$

$$z = (\ln|1+x| + k)\cdot(1+x)$$

$$\underline{\underline{y = \frac{1}{(\ln|1+x| + k)\cdot(1+x)}}}$$

Príklad č.27:

$$xy' - 4y = x^2\sqrt{y} \qquad \Big/ \cdot y^{-\frac{1}{2}}$$

$$xy^{-\frac{1}{2}}y' - 4y^{\frac{1}{2}} = x^2$$

$$z = y^{\frac{1}{2}}$$

$$z' = \frac{1}{2}y^{-\frac{1}{2}}y' \Rightarrow y' = 2y^{\frac{1}{2}}z'$$

$$xy^{-\frac{1}{2}}2z'y^{\frac{1}{2}} - 4z = x^2$$

$$2xz' - 4z = x^2$$

$$z' - \frac{2z}{x} = \frac{1}{2}x$$

Ďalej riešime homogénnu diferenciálnu rovnicu, t.j. bez pravej strany:

$$z' - \frac{2z}{x} = 0$$

$$\frac{z'}{z} - \frac{2}{x} = 0$$

$$\ln z = 2\ln x + \ln \alpha(x)$$

$$z = x^2\alpha(x)$$

$$z' = 2x\alpha(x) + x^2\alpha'(x)$$

Posledné dve rovnice dosadím do diferenciálnej rovnice s pravou stranou:

$$2x\alpha(x) + x^2\alpha'(x) - \frac{2x^2\alpha(x)}{x} = \frac{1}{2}x$$

$$x^2\alpha'(x) = \frac{1}{2}x$$

$$\alpha'(x) = \frac{1}{2x}$$

$$\alpha(x) = \frac{1}{2}\ln x + k$$

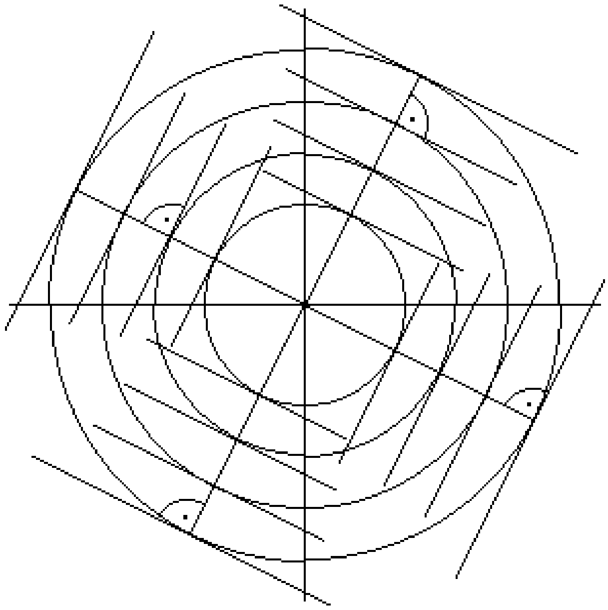
$$z = \left(\ln\sqrt{x} + k\right)x^2$$

$$y^{\frac{1}{2}} = \left(\ln\sqrt{x} + k\right)x^2$$

$$\underline{\underline{y = \left(\ln\sqrt{x} + k\right)^2 x^4}}$$

Ortogonálne trajektórie

- Hľadáme trajektórie, ktoré pretínajú krivky v uhle $\frac{\pi}{2}$



$$y' = \frac{-1}{y'_t}$$

$$F = F(x, y, z)$$

$$F_x^2 + F_y^2 \neq 0$$

$$F'_y(x, y, z) - F_x(x, y, z)y' = 0$$

$$F_x(x, y, z) = 0$$

$$x^2 + y^2 - \alpha^2 = 0 \quad ; \quad \alpha \neq 0$$

$$\left.\begin{aligned} F_x &= 2x \\ F_y &= 2y \end{aligned}\right\} 4x^2 + 4y^2 \neq 0 \quad \text{okrem bodu } [0,0]$$

$$2y - 2xy' = 0$$

$$y - xy' = 0$$

$$\frac{1}{x} - \frac{y'}{y} = 0$$

$$y = cx$$

Znižovanie rádu diferenciálnych rovníc

$$y^{(n)} = g(x)$$
$$y^{(n-1)} = \int g(x)dx + c_1$$
$$y^{(n-2)} = \int y^{(n-1)}dx + c_2 = \int\left(\int g(x)dx + c_1\right)dx + c_2$$
$$\vdots$$
$$y^{(n-i)} = \int y^{(n-i+1)}dx + c_i$$
$$\vdots$$
$$y = \int y'dx + c_n$$

Príklad č.28:

$$y'' = 6x - \frac{1}{x^2}$$
$$y' = \int\left(6x - \frac{1}{x^2}\right)dx + c_1$$
$$y' = 3x^2 + \frac{1}{x} + c_1$$
$$y = \int\left(3x^2 + \frac{1}{x} + c_1\right)dx + c_2$$
$$y = x^3 + \ln|x| + c_1 x + c_2$$

Príklad č.29:

$$y'' = x + \sin x$$
$$y' = \int(x + \sin x)dx + c_1$$
$$y' = \frac{1}{2}x^2 - \cos x + c_1$$
$$y = \int\left(\frac{1}{2}x^2 - \cos x + c_1\right)dx + c_2$$
$$y = \frac{1}{6}x^3 - \sin x + c_1 x + c_2$$

Príklad č.30:

$$y'' = 2x^3 \quad ; \qquad y(0) = 2 \;; \quad y'(0) = 1$$

$$y' = \int 2x^3 dx + c_1$$

$$y' = \frac{1}{2}x^4 + c_1$$

$$y = \frac{1}{10}x^5 + c_1 x + c_2$$

$$2 = \frac{1}{10}0^5 + c_1 0 + c_2$$

$$1 = \frac{1}{2}0^4 + c_1$$

$$c_1 = 1$$

$$c_2 = 2$$

$$y = \frac{1}{10}x^5 + x + 2$$

Príklad č.31:

$$xy'' - y' = 0$$

substitúcia : $y' = u \Rightarrow u' = y''$

$$xu' - u = 0$$

$$\frac{u'}{u} - \frac{1}{x} = 0$$

$$u = cx$$

$$y' = cx$$

$$y = \int cxdx + k$$

$$y = \frac{cx^2}{2} + k$$

Príklad č.32:

$$y'' - \sqrt{1 - y'^2} = 0$$

substitúcia : $y' = u \Rightarrow u' = y''$

$$u' - \sqrt{1 - u^2} = 0$$

$$\frac{u'}{\sqrt{1 - u^2}} - 1 = 0$$

$$\int \frac{u'}{\sqrt{1 - u^2}} du - \int 1 dx = 0$$

$$\arcsin u - x = c$$

$$\arcsin u = c + x$$

$$u = \sin(c + x)$$

$$y' = \sin(c + x)$$

$$y = \int \sin(c + x) dx$$

$$\underline{\underline{y = -\cos(c + x) + k}}$$

Príklad č.33:

$$y'' - \frac{y'}{x} = x^2$$

substitúcia : $y' = u \Rightarrow u' = y''$

$$u' - \frac{u}{x} = x^2$$

$$\frac{u'}{u} - \frac{1}{x} = 0$$

$$\ln u = \ln c + \ln x$$

$$u = cx \quad / \quad c = c(x)$$

$$u' = c'x + c$$

$$c'x + c - \frac{cx}{x} = x^2$$

$$c' = x$$

$$c = \frac{1}{2} x + k$$

$$u = (0{,}5x + k)x = y'$$

$$y = \int (0{,}5x^2 + kx) dx + q$$

$$\underline{\underline{y = \frac{1}{8} x^4 + \frac{1}{2} kx^2 + q}}$$

Príklad č.34: Zisti, či sú dané rovnice lineárne závislé: $y_1 = 3x - 7$; $y_2 = 2x + 5$

Použijeme Wronskyjan: $W = \begin{vmatrix} f(x_1) & f(x_2) \\ \dfrac{\partial f(x_1)}{\partial x_1} & \dfrac{\partial f(x_2)}{\partial x_2} \end{vmatrix}$

$$W = \begin{vmatrix} 3x-7 & 2x+5 \\ 3 & 2 \end{vmatrix}$$

$$W = 2(3x-7) - 3(2x+5)$$

$$W = -29$$

$\underline{W \neq 0 \;\Rightarrow\; \text{sú lineárne nezávislé.}}$

Príklad č.35: Nájdi lineárnu diferenciálnu rovnicu bez pravej strany, ak je daný fundamentálny systém riešení: $y_1 = 1$; $y_2 = \cos x$

$$W = \begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = -\sin x$$

$$\frac{1}{-\sin x} \cdot \begin{vmatrix} y'' & y' & y \\ 0 & 0 & 1 \\ -\cos x & -\sin x & \cos x \end{vmatrix} = 0$$

$$-\frac{1}{\sin x}\left(-y'\cos x + y''\sin x\right) = 0$$

$$y'\operatorname{cotg} x - y'' = 0$$

$$\underline{\underline{y'' - y'\operatorname{cotg} x = 0}}$$

Príklad č.36: Nájdi lineárnu diferenciálnu rovnicu bez pravej strany. Je daný fundamentálny systém riešenia: $y_1 = x$; $y_2 = e^x$

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1)$$

$$\frac{-1}{e^x(x-1)} \begin{vmatrix} y'' & y' & y \\ 0 & 1 & x \\ e^x & e^x & e^x \end{vmatrix} = 0$$

$$\frac{-1}{e^x(x-1)}\left(y''e^x + y'xe^x - y''xe^x - ye^x\right) = 0$$

$$\frac{-1}{x-1}\left(y''(x-1) - y'x + y\right) = 0$$

$$\underline{\underline{y'' - y'\frac{x}{x-1} + y\frac{1}{x-1} = 0}}$$

Fundamentálny systém

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \mathbf{K} + p_{n-1}(x)y' + p_n(x)y = 0$$

$g(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 + \mathbf{K} + c_{n-1} y_{n-1} + c_n y_n$ - fundamentálny systém.

S pravou stranou:

$$g(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 + \mathbf{K} + c_{n-1} y_{n-1} + c_n y_n + \chi$$

$$\chi = \sum_{i=1}^{n} y_i \int \frac{W_i}{W}$$

Príklad č.37:

$$y'' - \frac{2x}{1+x^2} y' + \frac{2y}{1+x^2} = 0 \quad y_1 = x$$

$$y_2 = y_1 \int z(x)dx$$

$$y_2 = x \int z(x)dx \Rightarrow \quad y_2' = \int zdx + xz$$

$$y_2'' = z + z + xz' = 2z + xz'$$

$$xz' + 2z - \frac{2x}{1+x^2} xz - \frac{2x}{1+x^2} \int zdx + \frac{2x}{1+x^2} \int zdx = 0$$

$$xz' + 2z - \frac{2x}{1+x^2} xz = 0$$

$$\frac{z'}{z} + \frac{2}{x} - \frac{2x}{1+x^2} = 0$$

$$\ln z + 2\ln x - \ln\left|1+x^2\right| = 0$$

$$z = \frac{x^2}{1+x^2} = \frac{1}{x^2} + 1$$

$$y_2 = x\int\left(\frac{1}{x^2} + 1\right)dx = x\left(\frac{-1}{x} + x\right)$$

$$y_2 = x^2 - 1$$

$$\underline{\underline{y = c_1 x + c_2\left(x^2 - 1\right)}}$$

Príklad č.38:

$$y'' - \frac{1+x}{x}y' - 2\frac{x-1}{x}y = 0 \qquad y_1 = e^{2x}$$

$$y_2 = e^{2x}\int z dx \Rightarrow \qquad y_2' = 2e^{2x}\int z\cdot dx + z\cdot e^{2x}$$

$$y_2'' = 4e^{2x}\int z\cdot dx + 4\cdot z\cdot e^{2x} + z'\cdot e^{2x}$$

$$e^{2x}\left(4\int z\cdot dx + 4z + z'\right) - \frac{1+x}{x}e^{2x}\left(2\int z\cdot dx + z\right) - 2\frac{x-1}{x}e^{2x}\int z\cdot dx = 0$$

$$4\int z\cdot dx + 4z + z' - \frac{1+x}{x}\left(2\int z\cdot dx + z\right) - 2\frac{x-1}{x}\int z\cdot dx = 0$$

$$\int z\cdot dx\cdot\left(4 - \frac{2}{x} - 2 + \frac{2}{x} - 2\right) + 4z + z' - \frac{1+x}{x}z = 0$$

$$\left(4 - \frac{1+x}{x}\right)\cdot z + z' = 0$$

$$\frac{z'}{z} + \frac{3x-1}{x} = 0$$

$$\frac{z'}{z} = \frac{1-3x}{x} = \frac{1}{x} - 3$$

$$\ln z = \ln x - 3x$$

$$\ln z = \ln x - \ln e^{3x} \Rightarrow z = \frac{x}{e^{3x}}$$

$$y_2 = e^{2x}\int \frac{x}{e^{3x}}dx = \begin{vmatrix} u = x & v' = e^{-3x} \\ u' = 1 & v = -\frac{1}{3}e^{-3x} \end{vmatrix} = e^{2x}\left(-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right)$$

$$y_2 = e^{2x}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{3}\frac{1}{3}e^{-3x}\right) = \frac{-1}{3e^x}\left(x + \frac{1}{3}\right)$$

$$\underline{\underline{y = c_1e^{2x} - c_2\frac{1}{3e^x}\left(x + \frac{1}{3}\right)}}$$

Príklad č.39:

Riešte diferenciálnu rovnicu s pravou stranou $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = x-1$, ak je dané riešenie bez pravej strany: $y_1 = x$

$y_2 = e^x$

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = (x-1)e^x \qquad W_1 = \begin{vmatrix} 0 & e^x \\ x-1 & e^x \end{vmatrix} = -(x-1)e^x \qquad W_2 = \begin{vmatrix} x & 0 \\ 1 & x-1 \end{vmatrix} = (x-1)x$$

$$\chi_1 = y_1 \int \frac{W_1}{W} dx = x \int \frac{-(x-1)e^x}{(x-1)e^x} dx = x \int -1 dx = -x^2$$

$$\chi_2 = y_2 \int \frac{W_2}{W} dx = e^x \int \frac{x(x-1)}{(x-1)e^x} dx = e^x \int \frac{x}{e^x} dx = \begin{vmatrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{vmatrix} =$$

$$= e^x\left(-xe^{-x} - \int e^{-x} dx\right) = e^x\left(-xe^{-x} + e^{-x}\right) = 1 - x$$

$$y = c_1 y_1 + c_2 y_2 + \chi_1 + \chi_2$$

$$\underline{\underline{y = c_1 x + c_2 e^x + 1 - x - x^2}}$$

Diferenciálne rovnice s konštantnými koeficientmi

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0\,; \quad a_i \in \Re$$
$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \ldots + a_{n-1} r + a_n = 0$$
$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_{n-1} e^{r_{n-1} x} + c_n e^{r_n x} = \sum_{i=1}^{n} \left[c_i \exp(r_i x) \right]$$

Príklad č.40:

$$y'' - 6y' = 0 \Rightarrow r^2 - 6r = 0$$
$$r_1 = 0 \wedge r_2 = 6$$
$$y = c_1 e^{0x} + c_2 e^{6x}$$
$$\underline{\underline{y = c_1 + c_2 e^{6x}}}$$

Príklad č.41:

$$y''' - 5y'' + 8y' - 4y = 0$$
$$r^3 - 5r^2 + 8r - 4 = 0$$

	1	−5	8	−4
1		1	−4	4
	1	−4	4	0

$$r_1 = 1 \Rightarrow (r-1)\left(r^2 - 4r + r\right) = 0$$
$$(r-1)(r-2)^2 = 0$$
$$(r-1)(r-2)(r-2) = 0$$

Pozor! Koreň $r_{2,3} = 2$ je dvojnásobný koreň!

$$\underline{\underline{y = c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x}}}$$

Príklad č.42:

$$y'' + 2y' + 4y = 0$$
$$r^2 + 2r + 4 \Rightarrow r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 4}}{2} = \underline{-1 \pm i\sqrt{3}}$$

Riešenie je komplexný koreň $\Rightarrow \underline{\underline{y = c_1 e^{-x} \sin\sqrt{3}x + c_2 e^{-x} \cos\sqrt{3}x}}$

Príklad č.43:

$$y''' - 3y'' + 3y' - y = 0$$
$$r^3 - 3r^2 + 3r - 1 = 0$$
$$(r-1)^3 = 0$$
$r_1 = 1 \wedge r_2 = 1 \wedge r_3 = 1 \Rightarrow$ trojnásobný koreň
$$\underline{\underline{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x}}$$

Príklad č.44:

$$y^{(4)} - 2y'' + y = 0$$
$$r^4 - 2r^2 + 1 = 0$$
$$(r^2 - 1)^2 = 0$$
$$r_1 = 1 \wedge r_2 = 1 \wedge r_3 = -1 \wedge r_4 = -1$$
$$\underline{\underline{y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}}}$$

Príklad č.45:

$$y''' - 4y'' + 5y' = 0$$
$$r^3 - 4r^2 + 5r = 0$$
$$r(r^2 - 4r + 5) = 0 \Rightarrow r = 0 \wedge r^2 - 4r + 5 = 0$$
$$r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$
$$\underline{r_1 = 0 \wedge r_{2,3} = 2 \pm i}$$
$$\underline{\underline{y = c_1 + c_2 e^{2x} \sin x + c_3 e^{2x} \cos x}}$$

Príklad č.46:

$$y^{(8)} - 4y^{(6)} + 4y^{(4)} = 0 \Rightarrow r^8 - 4r^6 + 4r^4 = 0 \Rightarrow r^4 (r^4 - 4r^2 + 4) = 0$$
$$r^4 = 0 \wedge r^4 - 4r^2 + 4 = 0 \Rightarrow r_1 = r_2 = r_3 = r_4 = \underline{0}$$
$$r_{5,6}^2 = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = 2 \Rightarrow r_5 = r_6 = \underline{\sqrt{2}} \wedge r_7 = r_8 = \underline{-\sqrt{2}}$$
$$\underline{\underline{y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{\sqrt{2}} + c_6 x e^{\sqrt{2}} + c_7 e^{-\sqrt{2}} + c_8 x e^{-\sqrt{2}}}}$$

Príklad č.47:

$$y^{(4)}+6y''+9y=0\Rightarrow r^4+6r^2+9=0\Rightarrow r_{1,2}^2=\frac{-6\pm\sqrt{36-4\cdot 9}}{2}=-3$$

$$r_{1,2}=\underline{\sqrt{3}\cdot i}\wedge r_{3,4}=\underline{-\sqrt{3}\cdot i}$$

$$\underline{\underline{y=c_1\cos\sqrt{3}x+c_2x\cos\sqrt{3}x+c_3\sin\sqrt{3}x+c_4x\sin\sqrt{3}x}}$$

Príklad č.48:

$$y''-2y'+5y=0\Rightarrow r^2-2r+5=0\Rightarrow r_{1,2}=\frac{2\pm\sqrt{4-20}}{2}=1\pm 2\cdot i$$

$$r_{1,2}=\underline{\sqrt{3}\cdot i}\wedge r_{3,4}=\underline{-\sqrt{3}\cdot i}$$

$$\underline{\underline{y=e^xc_1\cos 2x+e^xc_2\sin 2x}}$$

Príklad č.49: Riešte danú diferenciálnu rovnicu $y''-7y'+10y=A$, kde A je pravá strana danej diferenciálnej rovnice a je zastúpená danými funkciami:

a) $A=-12e^{3x}$

b) $A=20x^2-28x+14$

c) $A=-e^{2x}\cdot(6x+7)$

d) $A=65\sin 2x$

Najprv riešime danú diferenciálnu rovnicu bez pravej strany (homogénnu):

$$y''-7y'+10y=0$$

$$r^2-7r+10=0$$

$$r_{1,2}=\frac{7\pm\sqrt{49-40}}{2}$$

$$\underline{r_1=5\wedge r_2=2}$$

$$\underline{\underline{y_h=c_1e^{5x}+c_2e^{2x}}}$$

Riešenie s pravou stranou hľadáme ako súčet riešenia homogénnej diferenciálnej rovnice a partikulárneho riešenia: $y=y_h+y_p$

a) $y_p=ae^{3x}\Rightarrow y_p'=3ae^{3x}\Rightarrow y_p''=9ae^{3x}$

$$y''-7y'+10y=A$$

$$9ae^{3x}-21ae^{3x}+10ae^{3x}=-12e^{3x}$$

$$9a-21a+10a=-12\Rightarrow \underline{a=6}\Rightarrow \underline{\underline{y_p=6e^{3x}}}$$

Riešenie je teda: $\underline{\underline{y=c_1e^{5x}+c_2e^{2x}+6e^{3x}}}$

b) $y_p = ax^2 + bx + c \Rightarrow y'_p = 2ax + b \Rightarrow y''_p = 2a$

$$2a - 7\cdot(2ax+b) + 10\cdot(ax^2+bx+c) = 20x^2 - 28x + 14$$

$$2a - 14ax - 7b + 10ax^2 + 10bx + 10c = 20x^2 - 28x + 14$$

$$10a = 20$$

$$-14a + 10b = -28$$

$$\underline{2a - 7b + 10c = 14}$$

$$\underline{\underline{a = 2,\ \ b = 0,\ \ c = 1}} \Rightarrow y_p = 2x^2 + 1$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{5x} + c_2 e^{2x} + 2x^2 + 1}}$

c) $y_p = (ax+b)xe^{2x}$, pretože číslo 2 je koreňom homogénnej rovnice

$$y_p = (ax^2 + bx)\cdot e^{2x}$$

$$y'_p = (2ax+b)\cdot e^{2x} + 2\cdot(ax^2+bx)\cdot e^{2x}$$

$$y''_p = 2ae^{2x} + 2\cdot(2ax+b)\cdot e^{2x} + 2\cdot(2ax+b)\cdot e^{2x} + 4\cdot(ax^2+bx)\cdot e^{2x} =$$

$$= 2ae^{2x} + (8ax+4b)\cdot e^{2x} + (4ax^2+4bx)\cdot e^{2x}$$

$$2ae^{2x} + (8ax+4b)e^{2x} + (4ax^2+4bx)e^{2x} - (14ax+7b)e^{2x} - (14ax^2+14bx)e^{2x} +$$
$$(10ax^2+10bx)\cdot e^{2x} = (-6x-7)e^{2x}$$

$$2a + 8ax + 4b + 4ax^2 + 4bx - 14ax - 7b - 14ax^2 - 14bx + 10ax^2 + 10bx = -6x - 7$$

$$2a - 6ax - 3b = -6x - 7$$

$$-6a = -6 \Rightarrow \underline{a = 1}$$

$$\underline{2a - 3b = -7} \Rightarrow \underline{b = 3}$$

$$\underline{\underline{y_p = (x^2 + 3x)\cdot e^{2x}}}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{5x} + c_2 e^{2x} + (x+3)\cdot x\cdot e^{2x}}}$

d) $y_p = a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}}$, pretože funkcia sínus je imaginárnou zložkou komplexného čísla

$$y'_p = 2\cdot i\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}} \Rightarrow y''_p = -4\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}}$$

$$-4\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}} - 14\cdot i\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}} + 10\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}} = 65\cdot e^{2\cdot i\cdot x_{\mathrm{Im}}}$$

$$-4a - 14ia + 10a = 65$$

$$6a - 14ia = 65 \Rightarrow a = \frac{65}{6-14i}\cdot\frac{6+14i}{6+14i} = \frac{390+910i}{36+196} = \frac{390+910i}{232}$$

$$\mathrm{Im}\, y_p = \mathrm{Im}\left[\left(\frac{390}{232} + \frac{910}{232}i\right)\cdot(\cos 2x + i\cdot\sin 2x)\right] = \frac{390}{232}\cdot\sin 2x + \frac{910}{232}\cdot\cos 2x$$

$$\underline{\underline{\mathrm{Im}\, y_p = \frac{130}{232}(3\sin 2x + 7\cos 2x)}}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{5x} + c_2 e^{2x} + \frac{65}{116}(3\sin 2x + 7\cos 2x)}}$

Príklad č.50:

$$y'' - 2y' + 10y = 37\cos 3x = 37\cos 3x e^{0x}$$

$$r^2 - 2r + 10 = 0 \Rightarrow r_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = \underline{1 \pm 3i}$$

$$\underline{\underline{y_h = c_1 e^x \cos 3x + c_2 e^x \sin 3x}}$$

$$y_p = ae^{3ix_{\text{Re}}} \Rightarrow y_p' = 3iae^{3ix_{\text{Re}}} \Rightarrow y_p'' = -9ae^{3ix_{\text{Re}}}$$

$$-9ae^{3ix_{\text{Re}}} - 6iae^{3ix_{\text{Re}}} + 10ae^{3ix_{\text{Re}}} = 37e^{3ix_{\text{Re}}}$$

$$a - 6ia = 37$$

$$a = \frac{37}{1-6i} \cdot \frac{1+6i}{1+6i} = \frac{37+222i}{37} = 1 + 6i$$

$$y_p = \text{Re}\left[(1 + 6 \cdot i) \cdot (\cos 3x + i \cdot \sin 3x)\right]$$

$$y_p = \cos 3x + -6 \cdot \sin 3x$$

$$y = y_h + y_p$$

$$\underline{\underline{y = c_1 e^x \cos 3x + c_2 e^x \sin 3x + \cos 3x + -6 \cdot \sin 3x}}$$

Príklad č.51:

$$y'' - 7y' + 10y = 8e^{2x} \sin x$$

$$r^2 - 7r + 10 = 0 \Rightarrow r_{1,2} = \frac{7 \pm \sqrt{49-40}}{2} \Rightarrow \underline{r_1 = 5 \wedge r_2 = 2}$$

$$\underline{\underline{y_h = c_1 e^{5x} + c_2 e^{2x}}}$$

$$y_p = a \cdot e^{x\cdot(2+i)_{\text{Im}}} \Rightarrow y_p = (2+i) \cdot a \cdot e^{x\cdot(2+i)_{\text{Im}}} \Rightarrow y_p = (2+i)^2 \cdot a \cdot e^{x\cdot(2+i)_{\text{Im}}}$$

$$(2+i)^2 \cdot a \cdot e^{x\cdot(2+i)_{\text{Im}}} - 7(2+i) \cdot a \cdot e^{x\cdot(2+i)_{\text{Im}}} + 10 \cdot a \cdot e^{x\cdot(2+i)_{\text{Im}}} = 8 \cdot e^{(2x+i)_{\text{Im}}}$$

$$(2+i)^2 \cdot a - 7(2+i) \cdot a + 10 \cdot a = 8$$

$$-a - 3ia = 8$$

$$a = \frac{8}{-1-3i} \cdot \frac{-1+3i}{-1+3i} = -\frac{4}{5} + \frac{12}{5}i$$

$$y_p = \text{Im}\left[\left(-\frac{4}{5} + \frac{12}{5}i\right) \cdot (\cos x + i \cdot \sin x) e^{2x}\right]$$

$$y_p = e^{2x} \cdot \left(-\frac{4}{5}\sin x + \frac{12}{5}\cos x\right) = \underline{\frac{4}{5} \cdot e^{2x} \cdot (3 \cdot \cos x - \sin x)}$$

$$\underline{\underline{y = c_1 e^{5x} + c_2 e^{2x} + \frac{4}{5} \cdot e^{2x} \cdot (3 \cdot \cos x - \sin x)}}$$

Princíp superpozície

Príklad č.52:

$$y'' - 6y' + 9y = 3x - 8e^{x}$$

$$r^2 - 6r + 9 = 0 \Rightarrow r_{1,2} = \frac{6 \pm \sqrt{36-36}}{2} = 3 \Rightarrow \underline{y_h = c_1 e^{3x} + c_2 x e^{3x}}$$

$$y_{p1} = ax + b \Rightarrow y'_{p1} = a \Rightarrow y''_{p1} = 0$$

$$0 - 6a + 9 \cdot (ax + b) = 3x$$

$$9ax + 9b - 6a = 3x$$

$$9a = 3 \Rightarrow a = \frac{1}{3}$$

$$9b - 6a = 0 \Rightarrow 9b - 6\frac{1}{3} = 0 \Rightarrow b = \frac{2}{9}$$

$$\underline{y_{p1} = \frac{1}{3}x + \frac{2}{9}}$$

$$y_{p2} = ae^{x} \Rightarrow y'_{p2} = ae^{x} \Rightarrow y''_{p2} = ae^{x}$$

$$ae^{x} - 6ae^{x} + 9ae^{x} = -8e^{x}$$

$$4a = -8 \Rightarrow a = -2$$

$$\underline{y_{p2} = -2e^{x}}$$

$$y = y_h + y_{p1} + y_{p2} \Rightarrow \underline{\underline{y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{3}x + \frac{2}{9} - 2e^{x}}}$$

Príklad č.53:

$$y'' + y' - 6y = x + e^{x}$$

$$r^2 + r - 6 = 0 \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} \Rightarrow \underline{r_1 = -3 \wedge r_2 = 2}$$

$$\underline{y_h = c_1 e^{-3x} + c_2 e^{2x}}$$

$$y_{p1} = ax + b \Rightarrow y'_{p1} = a \Rightarrow y''_{p1} = 0$$

$$0 + a - 6 \cdot (ax + b) = x$$

$$a - 6b - 6ax = x$$

$$-6a = 1 \Rightarrow a = -\frac{1}{6}$$

$$a - 6b = 0 \Rightarrow b = -\frac{1}{36}$$

$$\underline{y_{p1} = -\frac{1}{6}x - \frac{1}{36}}$$

$$y_{p2} = ae^{x} \Rightarrow y'_{p2} = ae^{x} \Rightarrow y''_{p2} = ae^{x}$$

$$ae^{x} + ae^{x} - 6ae^{x} = e^{x}$$

$$-4a = 1$$

$$a = -\frac{1}{4}$$

$$\underline{y_{p2} = -\frac{1}{4}e^{x}}$$

$$y = y_h + y_{p1} + y_{p2} \Rightarrow \underline{\underline{y = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{6}x - \frac{1}{36} - \frac{1}{4}e^{x}}}$$

Príklad č.54:

$$y'' - 2y' + y = \frac{e^x}{x}$$

$$r^2 - 2r + 1 = 0 \Rightarrow \underline{r_{1,2} = 1} \Rightarrow \underline{y_h = c_1 e^x + c_2 x e^x} \Rightarrow f_1(x) = e^x, \quad f_2(x) = xe^x, \quad g(x) = \frac{e^x}{x}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}(1+x) - e^{2x}x = e^{2x} \qquad W = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x + xe^x \end{vmatrix} = -xe^x \frac{e^x}{x} = -e^{2x} \qquad W_1 = \begin{vmatrix} 0 & f_2(x) \\ g(x) & f_2'(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x} \qquad W_2 = \begin{vmatrix} f_1(x) & 0 \\ f_1'(x) & g(x) \end{vmatrix}$$

$$\psi_1 = y_{p1} = e^x \int \frac{-e^{2x}}{e^{2x}} dx = e^x \int -1 \cdot dx = -xe^x \qquad \psi_1 = f_1(x) \cdot \int \frac{W_1}{W} dx$$

$$\psi_2 = y_{p2} = xe^x \int \frac{\frac{1}{x} e^{2x}}{e^{2x}} dx = xe^x \int \frac{1}{x} dx = xe^x \ln x \qquad \psi_2 = f_2(x) \cdot \int \frac{W_2}{W} dx$$

$$\underline{\underline{y = c_1 e^x + c_2 xe^x - xe^x + xe^x \ln x}}$$

Príklad č.55:

$$y'' + y = \frac{1}{\sin x}$$

$$r^2 + 1 = 0 \Rightarrow \underline{r_{1,2} = \pm i} \Rightarrow \underline{y_h = c_1 \cos x + c_2 \sin x}$$

$$f_1(x) = \cos x, \quad f_2(x) = \sin x, \quad g(x) = \frac{1}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \qquad W = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix} = -1 \qquad W_1 = \begin{vmatrix} 0 & f_2(x) \\ g(x) & f_2'(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin x} \end{vmatrix} = \frac{\cos x}{\sin x} \qquad W_2 = \begin{vmatrix} f_1(x) & 0 \\ f_1'(x) & g(x) \end{vmatrix}$$

$$\psi_1 = y_{p1} = \cos x \int \frac{-1}{1} dx = -x \cos x \qquad \psi_1 = f_1(x) \cdot \int \frac{W_1}{W} dx$$

$$\psi_2 = y_{p2} = \sin x \int \frac{\cos x}{\sin x} dx = \sin x \ln|\sin x| \qquad \psi_2 = f_2(x) \cdot \int \frac{W_2}{W} dx$$

$$\underline{\underline{y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|}}$$

Príklad č.56: Riešte danú diferenciálnu rovnicu $y'' - 6y' + 9y = A$, kde A je pravá strana danej diferenciálnej rovnice a je zastúpená danými funkciami:

a)	$A = x^2 e^{4x}$	d)	$A = x\cos 2x$
b)	$A = e^x \sin 3x$	e)	$A = x^2 e^{3x}$
c)	$A = xe^{3x}$	f)	$A = 2\sin x$

Najprv riešime danú diferenciálnu rovnicu bez pravej strany (homogénnu):

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0 \Rightarrow \underline{r_{1,2} = 3}$$

$$\underline{\underline{y_h = c_1 e^{3x} + c_2 x e^{3x}}}$$

Riešenie s pravou stranou hľadáme ako súčet riešenia homogénnej diferenciálnej rovnice a partikulárneho riešenia: $y = y_h + y_p$

a) $y_p = (ax^2 + bx + c)\cdot e^{4x}$

$$y_p' = (2ax + b)\cdot e^{4x} + 4\cdot(ax^2 + bx + c)\cdot e^{4x}$$

$$y_p'' = 2ae^{4x} + 4\cdot(2ax + b)\cdot e^{4x} + 4\cdot(2ax + b)\cdot e^{4x} + 16\cdot(ax^2 + bx + c)\cdot e^{4x}$$

$$y_p'' = 2ae^{4x} + 8\cdot(2ax + b)\cdot e^{4x} + 16\cdot(ax^2 + bx + c)\cdot e^{4x}$$

$$2ae^{4x} + 8\cdot(2ax + b)\cdot e^{4x} + 16\cdot(ax^2 + bx + c)\cdot e^{4x} -$$

$$-6\cdot\left[(2ax + b)\cdot e^{4x} + 4\cdot(ax^2 + bx + c)\cdot e^{4x}\right] + 9\cdot(ax^2 + bx + c)\cdot e^{4x} = x^2 e^{4x}$$

$$ax^2 + 4ax + bx + 2a + 2b + c = x^2$$

$$a = 1$$

$$4a + b = 0 \qquad \Rightarrow b = -4$$

$$\underline{2a + 2b + c = 0} \Rightarrow c = 6$$

$$\underline{y_p = (x^2 - 4x + 6)\cdot e^{4x}}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{3x} + c_2 x e^{3x} + (x^2 - 4x + 6)\cdot e^{4x}}}$

b) $y_p = a\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}}$

$$y_p' = a\cdot(1 + 3i)\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}}$$

$$\underline{y_p'' = a\cdot(1 + 3i)^2\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}}}$$

$$a\cdot(1 + 3i)^2\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}} - 6\cdot a\cdot(1 + 3i)\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}} + 9\cdot a\cdot e^{x\cdot(1+3i)_{\mathrm{Im}}} = e^{x\cdot(1+3i)_{\mathrm{Im}}}$$

$$a\cdot(1 + 6i - 9) - a\cdot(6 + 18i) + 9\cdot a = 1$$

$$a\cdot(-8 + 6i - 6 - 18i + 9) = 1$$

$$a\cdot(-5 - 12i) = 1 \Rightarrow a = \frac{1}{-5 - 12i}\cdot\frac{-5 + 12i}{-5 + 12i} = -\frac{5}{169} + \frac{12}{169}i$$

$$y_p = \mathrm{Im}\left[\left(-\frac{5}{169} + \frac{12}{169}i\right)\cdot(\cos 3x + i\cdot\sin 3x)\cdot e^x\right] = \underline{\frac{1}{169}e^x(12\cos 3x - 5\sin 3x)}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{169}e^x(12\cos 3x - 5\sin 3x)}}$

c) $y_p = (ax+b)\cdot x^2 \cdot e^{3x}$, pretože 3 je v tomto prípade trojnásobný koreň

$$y_p = (ax^3 + bx^2)\cdot e^{3x}$$
$$y_p' = (3ax^2 + 2bx)\cdot e^{3x} + 3\cdot(ax^3 + bx^2)\cdot e^{3x}$$
$$y_p'' = (6ax+2b)\cdot e^{3x} + 3\cdot(3ax^2+2bx)\cdot e^{3x} + 3\cdot(3ax^2+2bx)\cdot e^{3x} + 9\cdot(ax^3+bx^2)\cdot e^{3x}$$
$$y_p'' = 9\cdot(ax^3+bx^2)\cdot e^{3x} + 6\cdot(3ax^2+2bx)\cdot e^{3x} + (6ax+2b)\cdot e^{3x}$$

$$9\cdot(ax^3+bx^2)\cdot e^{3x} + 6\cdot(3ax^2+2bx)\cdot e^{3x} + (6ax+2b)\cdot e^{3x} - 6\cdot(3ax^2+2bx)\cdot e^{3x} + -18\cdot(ax^3+bx^2)\cdot e^{3x} + 9\cdot(ax^3+bx^2)\cdot e^{3x} = x\cdot e^{3x}$$

$$6ax + 2b = x$$
$$6a = 1 \Rightarrow a = \frac{1}{6}$$
$$2b = 0 \Rightarrow b = 0$$
$$y_p = \frac{1}{6}x^3 e^{3x}$$

Riešenie je teda: $y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{6}x^3 e^{3x}$

d) $y_p = (ax+b)\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}}$

$$y_p' = a\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}} + 2\cdot i\cdot(ax+b)\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}}$$
$$y_p'' = 2\cdot i\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}} + 2\cdot i\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}} - 4\cdot(ax+b)\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}}$$
$$y_p'' = 4\cdot i\cdot a\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}} - 4\cdot(ax+b)\cdot e^{2\cdot i\cdot x_{\mathrm{Re}}}$$

$$4\cdot i\cdot a - 4\cdot(ax+b) - 6\cdot[a + 2\cdot i\cdot(ax+b)] + 9\cdot(ax+b) = x$$
$$4ai - 4ax - 4b - 6a - 12axi - 12bi + 9ax + 9b = x$$
$$5a - 12ai = 1 \Rightarrow a = \frac{1}{5-12i}\cdot\frac{5+12i}{5+12i} = \frac{5}{169} + \frac{12}{169}i$$
$$5b - 12bi - 6a + 4ai = 0 \Rightarrow b\cdot(5-12i) + a\cdot(-6+4i) = 0$$

$$b\cdot(5-12i) = \left(\frac{5}{169} + \frac{12}{169}i\right)\cdot(6-4i) = \frac{1}{169}(30 - 20i + 72i + 48) = \frac{1}{169}(78+52i)$$
$$b = \frac{1}{169}\frac{78+52i}{5-12i}\cdot\frac{5+12i}{5+12i} = \frac{1}{169}\cdot\left(-\frac{234}{169} + \frac{1196}{169}i\right)$$
$$y_p = \mathrm{Re}\left\{\left[\frac{1}{169}\cdot(78+52i)x + \frac{1}{169}\cdot\left(-\frac{234}{169} + \frac{1196}{169}i\right)\right]\cdot(\cos 2x + i\cdot\sin 2x)\right\}$$
$$y_p = \frac{13182x - 234}{169^2}\cos 3x - \frac{8788x + 1192}{169^2}\sin 3x$$

Riešenie je teda: $y = c_1 e^{3x} + c_2 x e^{3x} + \frac{13182x - 234}{169^2}\cos 3x - \frac{8788x + 1192}{169^2}\sin 3x$

e) $y_p = \left(ax^2 + bx + c\right)\cdot x^2 \cdot e^{3x}$, pretože 3 je v tomto prípade trojnásobný koreň

$$y_p = \left(ax^4 + bx^3 + cx^2\right)\cdot e^{3x}$$

$$y'_p = \left(4ax^3 + 3bx^2 + 2cx\right)\cdot e^{3x} + 3\cdot\left(ax^4 + bx^3 + cx^2\right)\cdot e^{3x}$$

$$y''_p = \left(12ax^2 + 6bx + 2c\right)\cdot e^{3x} + 3\cdot\left(4ax^3 + 3bx^2 + 2cx\right)\cdot e^{3x} +$$

$$+ 3\cdot\left(4ax^3 + 3bx^2 + 2cx\right)\cdot e^{3x} + 9\cdot\left(ax^4 + bx^3 + cx^2\right)\cdot e^{3x}$$

$$\underline{y''_p = \left(12ax^2 + 6bx + 2c\right)\cdot e^{3x} + 6\cdot\left(4ax^3 + 3bx^2 + 2cx\right)\cdot e^{3x} + 9\cdot\left(ax^4 + bx^3 + cx^2\right)\cdot e^{3x}}$$

$$\left(12ax^2 + 6bx + 2c\right) + 6\cdot\left(4ax^3 + 3bx^2 + 2cx\right) + 9\cdot\left(ax^4 + bx^3 + cx^2\right) -$$

$$-6\cdot\left[\left(4ax^3 + 3bx^2 + 2cx\right) + 3\cdot\left(ax^4 + bx^3 + cx^2\right)\right] + 9\cdot\left(ax^4 + bx^3 + cx^2\right) = x^2$$

$$12ax^2 + 6bx + 2c = x^2$$

$$12a = 1 \Rightarrow a = \frac{1}{12}$$

$$6b = 0 \Rightarrow b = 0$$

$$2c = 0 \Rightarrow c = 0$$

$$\underline{y_p = \frac{1}{12}\cdot x^4 \cdot e^{3x}}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{12}\cdot x^4 \cdot e^{3x}}}$

f) $y_p = a \cdot e^{i\cdot x_{\mathrm{Im}}}$

$$y'_p = i \cdot a \cdot e^{i\cdot x_{\mathrm{Im}}}$$

$$\underline{y''_p = -a \cdot e^{i\cdot x_{\mathrm{Im}}}}$$

$$-a\cdot e^{i\cdot x_{\mathrm{Im}}} - 6\cdot i\cdot a\cdot e^{i\cdot x_{\mathrm{Im}}} + 9\cdot a\cdot e^{i\cdot x_{\mathrm{Im}}} = 2\cdot e^{i\cdot x_{\mathrm{Im}}}$$

$$-a - 6\cdot i\cdot a + 9\cdot a = 2$$

$$8a - 6ia = 2 \Rightarrow 4a - 3ia = 1$$

$$a = \frac{1}{4-3i}\cdot\frac{4+3i}{4+3i} = \frac{4}{25} + \frac{3}{25}i$$

$$y_p = \mathrm{Im}\left[\frac{1}{25}\cdot(4+3i)\cdot(\cos 3x + i\cdot\sin 3x)\right] = \underline{\frac{1}{25}\cdot(3\cos x + 4\sin x)}$$

Riešenie je teda: $\underline{\underline{y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{25}\cdot(3\cos x + 4\sin x)}}$

Systémy diferenciálnych rovníc

Eliminačná metóda

Príklad č.57:

$x' = y + 1;\quad x = x(t)$

$y' = x + 1;\quad y = y(t) \Rightarrow x = y' - 1 \Rightarrow x' = y''$

$y'' = y + 1 \Rightarrow y'' - y = 1$

Najskôr riešime homogénnu diferenciálnu rovnicu:

$r^2 - 1 = 0 \Rightarrow r = \pm 1$

$\underline{\underline{y_h = c_1 e^t + c_2 e^{-t}}}$

Riešenie partikulárne (riešenie s pravou stranou):

$y_p = a \Rightarrow y_p' = y_p'' = 0$

$0 - a = 1 \Rightarrow a = -1$

$\underline{\underline{y_p = -1}}$

$\underline{\underline{y = y_h + y_p = c_1 e^t + c_2 e^{-t} - 1}}$

Pre funkciu $x(t)$ potom platí: $x = \left(c_1 e^t + c_2 e^{-t} - 1\right)' - 1 \Rightarrow \underline{\underline{x = c_1 e^t - c_2 e^{-t} - 1}}$

Príklad č.58:

$x' = y + t$

$y' = x - t \Rightarrow x = y' + t \Rightarrow x' = y'' + 1$

$y'' + 1 = y + t \Rightarrow y'' - y = t - 1$

Homogénna diferenciálna rovnica: $y'' - y = 0$

$r^2 - 1 = 0 \Rightarrow r = \pm 1$

$\underline{\underline{y_h = c_1 e^t + c_2 e^{-t}}}$

Partikulárne riešenie: $y_p = at + b \Rightarrow y_p' = a \Rightarrow y_p'' = 0$

$0 - (at + b) = t - 1$

$-at - b = t - 1$

$-a = 1 \wedge -b = -1 \Rightarrow a = -1 \wedge b = 1$

$\underline{\underline{y_p = -t + 1}}$

Riešenie systému diferenciálnych rovníc: $\underline{\underline{y = c_1 e^t + c_2 e^{-t} - t + 1}}$

$x = y' + t \Rightarrow x = \left(c_1 e^t + c_2 e^{-t} - t + 1\right)' + t \Rightarrow \underline{\underline{x = c_1 e^t - c_2 e^{-t} + t - 1}}$

Príklad č.59:

$$4x' - y' + 3x = \sin t$$
$$x' + y = \cos t \Rightarrow y = \cos t - x' \Rightarrow y' = -\sin t - x''$$

$$4x' - (-\sin t - x'') + 3x = \sin t$$
$$4x' + x'' + 3x = 0$$

Homogénna diferenciálna rovnica: $x'' + 4x' + 3x = 0$

$$r^2 + 4r + 3 = 0 \Rightarrow r_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 3}}{2}$$
$$r_1 = -1 \wedge r_2 = -3$$
$$x = c_1 e^{-t} + c_2 e^{-3t}$$

$$y = \cos t - \left(c_1 e^{-t} + c_2 e^{-3t}\right)' = \cos t - \left(-c_1 e^{-t} - 3c_2 e^{-3t}\right)$$

Riešenie systému diferenciálnych rovníc: $y = \cos t + c_1 e^{-t} + 3c_2 e^{-3t}$.

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

Príklad č.60:

$$y'' - z = 0 \Rightarrow z = y'' \Rightarrow z'' = y^{(4)}$$
$$z'' - y = 0$$

$$y^{(4)} - y = 0$$

Homogénna diferenciálna rovnica: $y^{(4)} - y = 0$

$$r^4 - 1 = 0 \Rightarrow r^4 = 1 \Rightarrow r_{1,2} = \pm 1 \wedge r_{3,4} = \pm i$$
$$y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$z = \left(c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t\right)'' = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Riešenie systému diferenciálnych rovníc: $y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

$$z = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Príklad č.61:

$$x' = -x - 2y$$
$$y' = 3x + 4y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{vmatrix} = (-1-\lambda)\cdot(4-\lambda) - (-2)\cdot 3 = \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} \Rightarrow \underline{\lambda_1 = 1 \wedge \lambda_2 = 2}$$

kde $\lambda_{1,2}$ sú vlastné čísla matice

$$\begin{pmatrix} -1-\lambda_1 & -2 \\ 3 & 4-\lambda_1 \end{pmatrix} = \begin{pmatrix} -1-1 & -2 \\ 3 & 4-1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot e^t$$

$$\begin{pmatrix} -1-\lambda_2 & -2 \\ 3 & 4-\lambda_2 \end{pmatrix} = \begin{pmatrix} -1-2 & -2 \\ 3 & 4-2 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix} \Rightarrow \xi_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot e^{2t}$$

$$\begin{pmatrix} e^t & 2\cdot e^{2t} \\ -e^t & -3\cdot e^{2t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x = c_1 \cdot e^t + 2\cdot c_2 \cdot e^{2t} \\ y = -c_1 \cdot e^t - 3\cdot c_2 \cdot e^{2t} \end{array}$$

kde pre ξ_i musí platiť, že súčin ξ_i s prislúchajúcou maticou (na hlavnej diagonále odčítame vlastné číslo λ_i) sa rovná 0.

Príklad č.62:

$$\begin{aligned} x' &= 2x - y + z \\ y' &= x + 2y - z \\ z' &= x - y + 2z \end{aligned} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

vlastné čísla:

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 + 1 - 1 - (2-\lambda) - (2-\lambda) + (2-\lambda) = (2-\lambda)^3 - (2-\lambda)$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 2 + \lambda = 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$$

$$\underline{\lambda_1 = 1 \wedge \lambda_2 = 2 \wedge \lambda_3 = 3}$$

Riešenie vektorov ξ_i pre vlastné čísla matice $\lambda_i = 1, 2, 3$:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot e^t$$

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \Rightarrow \xi_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot e^{2t}$$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \Rightarrow \xi_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{3t}$$

$$\begin{pmatrix} 0 & e^{2t} & e^{3t} \\ e^t & e^{2t} & 0 \\ e^t & e^{2t} & e^{3t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} x &= c_2 \cdot e^{2t} + c_3 \cdot e^{3t} \\ y &= c_1 \cdot e^t + c_2 \cdot e^{2t} \\ z &= c_1 \cdot e^t + c_2 \cdot e^{2t} + c_3 \cdot e^{3t} \end{aligned}$$

Príklad č.63:

$$\begin{aligned} x' + x - 8y &= 0 \\ y' - x - y &= 0 \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

vlastné čísla: $\begin{vmatrix} -1-\lambda & 8 \\ 1 & 1-\lambda \end{vmatrix} = (-1-\lambda) \cdot (1-\lambda) - 8 \cdot 1 = \lambda^2 - 9 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 3}$

pre $\underline{\lambda_1 = 3}$: $\begin{pmatrix} -4 & 8 \\ 1 & -2 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot e^{3t}$ pre $\underline{\lambda_2 = -3}$: $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \cdot e^{-3t}$

$$\begin{pmatrix} 2 \cdot e^{3t} & 4 \cdot e^{-3t} \\ e^{3t} & -e^{-3t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{aligned} x &= 2 \cdot c_1 \cdot e^{3t} + 4 \cdot c_2 \cdot e^{-3t} \\ y &= c_1 \cdot e^{3t} - c_2 \cdot e^{-3t} \end{aligned}$$

Príklad č.64:

$$\begin{aligned} x' &= x + y \\ y' &= -2x + 3y \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

vlastné čísla: $\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \underline{\lambda_{1,2} = 2 \pm i}$

$$\lambda = 2 + i \Rightarrow \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \cdot \xi = 0 \Rightarrow \xi = \begin{pmatrix} 1-i \\ 2 \end{pmatrix} \cdot e^{(2+i)\cdot t} = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \cdot e^{it}$$

$$e^{it} = \cos t + i \cdot \sin t \Rightarrow \xi = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \cdot (\cos t + i \cdot \sin t)$$

ξ_1 – reálna zložka $\Rightarrow \xi_1 = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \sin t \right]$

ξ_2 – imaginárna zložka $\Rightarrow \xi_2 = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \cos t \right]$

Potom riešením bude:

$$\begin{pmatrix} e^{2t} \cdot \cos t + e^{2t} \cdot \sin t & e^{2t} \cdot \sin t - e^{2t} \cdot \cos t \\ 2 \cdot e^{2t} \cdot \cos t & 2 \cdot e^{2t} \cdot \sin t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = c_1 \cdot e^{2t} \cdot (\cos t + \sin t) + c_2 \cdot e^{2t} \cdot (\sin t - \cos t)$$

$$y = 2 \cdot c_1 \cdot e^{2t} \cdot \cos t + 2 \cdot c_2 \cdot e^{2t} \cdot \sin t$$

Príklad č.65:

$$\begin{aligned} x' &= x + y \\ y' &= -5x - y \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$ a vlastné čísla: $\begin{vmatrix} 1-\lambda & 1 \\ -5 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 2i}$

$$\lambda = 2i \Rightarrow \begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix} \cdot \xi = 0 \Rightarrow \xi = \begin{pmatrix} -1+2i \\ 1 \end{pmatrix} \cdot e^{2it} = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] \cdot e^{2it}$$

$$e^{2it} = \cos 2t + i \cdot \sin 2t \Rightarrow \xi = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] \cdot (\cos 2t + i \cdot \sin 2t)$$

$$\xi_1^{\mathrm{Re}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \sin 2t \qquad \xi_2^{\mathrm{Im}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \cos 2t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\cos 2t + 2\sin 2t & -\sin 2t + 2\cos 2t \\ \cos 2t & \sin 2t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x = c_1(2\sin 2t - \cos 2t) + c_2(2\cos 2t - \sin 2t)$$

$$y = c_1(\cos 2t) + c_2(\sin 2t)$$

Príklad č.66:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ a vlastné čísla: } \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \underline{\lambda_{1,2} = 2 \pm i}$$

$$\lambda = 2+i \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \cdot \xi = 0 \Rightarrow \xi = \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot e^{(2+i)\cdot t} = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \cdot e^{it}$$

$$e^{it} = \cos t + i \cdot \sin t \Rightarrow \xi = e^{2t} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \cdot (\cos t + i \cdot \sin t)$$

$$\xi_1^{\mathrm{Re}} = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \cos t - e^{2t} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \sin t \qquad \xi_2^{\mathrm{Im}} = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \sin t + e^{2t} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \cos t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{2t} \cdot \cos t & e^{2t} \cdot \sin t \\ e^{2t} \cdot \sin t & -e^{2t} \cdot \cos t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{matrix} x = c_1 \cdot e^{2t} \cdot \cos t + c_2 \cdot e^{2t} \cdot \sin t \\ y = c_1 \cdot e^{2t} \cdot \sin t - c_2 \cdot e^{2t} \cdot \cos t \end{matrix}$$

Príklad č.67:

$$\begin{matrix} x' = 2x + y \\ y' = x + 3y - z \\ \underline{z' = -x + 2y + 3z} \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \underline{\lambda_1 = 2 \wedge \lambda_{2,3} = 3 \pm i}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \cdot \xi_1 = 0 \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{2t}$$

$$\begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix} \cdot \xi = 0 \Rightarrow \xi = \begin{pmatrix} -1 \\ -1-i \\ -2+i \end{pmatrix} \cdot e^{(3+i)t} = e^{3t} \cdot \left[\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} + i \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right] \cdot (\cos t + i \cdot \sin t)$$

$$\xi_2^{\mathrm{Re}} = e^{3t} \cdot \left[\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \cdot \cos t - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \sin t \right] \qquad \xi_3^{\mathrm{Im}} = e^{3t} \cdot \left[\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \cdot \sin t + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \cos t \right]$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{2t} & -e^{3t} \cdot \cos t & -e^{3t} \cdot \sin t \\ 0 & e^{3t} \cdot (\sin t - \cos t) & -e^{3t} \cdot (\sin t + \cos t) \\ e^{2t} & e^{3t} \cdot (-2\cos t - \sin t) & e^{3t} \cdot (-2\sin t + \cos t) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Veierova teória

Pre trojnásobný koreň:

a) $\mu_1 = 3 \Rightarrow h_1 = 0;\ \ \sigma_1 = 3$

$(\mathbf{A} - \lambda \cdot \mathbf{E}) \cdot \kappa_{1j} = 0$

κ_{11}	κ_{12}	κ_{13}

– málo sa vyskytuje – výnimočný

b) $\mu_1 = 2 \Rightarrow h_1 = 1;\ \ \sigma_1 = 2$

$\mu_2 = 3 \Rightarrow \sigma_2 = 1$

$(\mathbf{A} - \lambda \cdot \mathbf{E})^2 \cdot \kappa_{21} = 0$

$(\mathbf{A} - \lambda \cdot \mathbf{E}) \cdot \kappa_{21} = \kappa_{11}$

$(\mathbf{A} - \lambda \cdot \mathbf{E}) \cdot \kappa_{12} = 0$

κ_{21}	
κ_{11}	κ_{12}

$\xi_1 = \kappa_{11} \cdot e^{\lambda t}$

$\xi_2 = \kappa_{12} \cdot e^{\lambda t}$

$\xi_3 = (\kappa_{11} \cdot t + \kappa_{12}) \cdot e^{\lambda t}$

kde λ je koreň (vlastné číslo matice), vektory κ_{ij} musia byť lineárne nezávislé

c) $\mu_1 = 1 \Rightarrow h_1 = 2;\ \ \sigma_1 = 1$

$\mu_2 = 2 \Rightarrow \sigma_2 = 1$

$\mu_3 = 3 \Rightarrow \sigma_3 = 1$

$(\mathbf{A} - \lambda \cdot \mathbf{E})^3 \cdot \kappa_{31} = 0$

$(\mathbf{A} - \lambda \cdot \mathbf{E}) \cdot \kappa_{31} = \kappa_{21}$

$(\mathbf{A} - \lambda \cdot \mathbf{E}) \cdot \kappa_{21} = \kappa_{11} = (\mathbf{A} - \lambda \cdot \mathbf{E})^2 \cdot \kappa_{31}$

κ_{31}
κ_{21}
κ_{11}

$\xi_1 = \kappa_{11} \cdot e^{\lambda t}$

$\xi_2 = (\kappa_{11} \cdot t + \kappa_{21}) \cdot e^{\lambda t}$

$\xi_3 = \left(\kappa_{11} \cdot \frac{t^2}{2} + \kappa_{21} \cdot t + \kappa_{31}\right) \cdot e^{\lambda t}$

Príklad č.68:

$$\begin{aligned} x' &= -4x + 2y + 5z \\ y' &= 6x - y - 6z \\ z' &= -8x + 3y + 9z \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -4 & 2 & 5 \\ 6 & -1 & -6 \\ -8 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{vmatrix} -4-\lambda & 2 & 5 \\ 6 & -1-\lambda & -6 \\ -8 & 3 & 9-\lambda \end{vmatrix} = 0 \Rightarrow \underline{\lambda_1 = 2 \wedge \lambda_{2,3} = 1}$$

Máme dvojnásobný koreň!

$$\lambda_1 = 2: \begin{pmatrix} -6 & 2 & 5 \\ 6 & -3 & -6 \\ -8 & 3 & 7 \end{pmatrix} \cdot \xi_1 = 0 \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot e^{2t}$$

$\lambda_{2,3} = 1: \begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix}$, zistíme počet lineárne nezávislých riadkov (stĺpcov):

$$\lambda_{2,3} = 1: \begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \Rightarrow h = 2; \quad \begin{aligned} &\mu = \text{rozmer - hodnosť} \\ &\mu_1 = 3 - 2 = 1 \\ &\sigma_1 = \mu_1 = 1 \end{aligned}$$

$$\begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix} \cdot \begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 \\ 6 & -2 & -6 \\ -6 & 2 & 6 \end{pmatrix} \Rightarrow \begin{aligned} &h_2 = 1 \\ &\mu_2 = 3 - 1 = 2 \\ &\sigma_2 = \mu_2 - \mu_1 = 2 - 1 = 1 \end{aligned}$$

$$\begin{pmatrix} -3 & 1 & 3 \\ 6 & -2 & -6 \\ -6 & 2 & 6 \end{pmatrix} \cdot \kappa_{21} = 0 \Rightarrow \kappa_{21} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

κ_{21}
κ_{11}

$$\kappa_{11} = \begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix} \cdot \kappa_{21} = \begin{pmatrix} -5 & 2 & 5 \\ 6 & -2 & -6 \\ -8 & 3 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^t \qquad \xi_3 = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot t + \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right] \cdot e^t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{2t} & e^t & (t+1) \cdot e^t \\ -2 \cdot e^{2t} & 0 & 3 \cdot e^t \\ 2 \cdot e^{2t} & e^t & t \cdot e^t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{aligned} x &= c_1 \cdot e^{2t} + c_2 \cdot e^t + c_3 \cdot (t+1) \cdot e^t \\ y &= -2 \cdot c_1 \cdot e^{2t} + 3 \cdot c_3 \cdot e^t \\ z &= 2 \cdot c_1 \cdot e^{2t} + c_2 \cdot e^t + c_3 \cdot t \cdot e^t \end{aligned}$$

Príklad č.69:

$$x' = 2x - 2y - 4z \qquad y' = 4y + 4z \qquad z' = -y$$

$$\begin{vmatrix} 2-r & -2 & -4 \\ 0 & 4-r & 4 \\ 0 & -1 & -r \end{vmatrix} = 0 \Rightarrow r_{1,2,3} = 2$$

$$(\mathbf{A} - r\mathbf{E}) = \begin{pmatrix} 0 & -2 & -4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix} \quad h_1 = 1,\ \mu_1 = 3 - 1 = 2,\ \sigma_1 = 2 = \mu_1$$

κ_{21}	
κ_{11}	κ_{12}

$$(\mathbf{A} - r\mathbf{E})^2 = 0 \Rightarrow \quad h_2 = 1,\ \mu_2 = 3,\ \sigma_2 = 1$$

$$\kappa_{21} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \kappa_{11} = \begin{pmatrix} 0 & -2 & -4 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \qquad \kappa_{12} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\xi_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \cdot e^{2t} \qquad \xi_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \cdot e^{2t} \qquad \xi_3 = \left[\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \cdot t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \cdot e^{2t}$$

$$\begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} -2e^{2t} & 0 & -2te^{2t} \\ 2e^{2t} & 2e^{2t} & (2t+1)e^{2t} \\ e^{2t} & -e^{2t} & te^{2t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{aligned} x &= -2c_1e^{2t} - 2tc_3e^{2t} \\ y &= 2c_1e^{2t} + 2c_2e^{2t} + (2t+1)c_3e^{2t} \\ z &= c_1e^{2t} - c_2e^{2t} + tc_3e^{2t} \end{aligned}$$

Príklad č.70:

$$x' = -2x + y - 2z$$
$$y' = x - 2y + 2z$$
$$z' = 3x - 3y + 5z$$

$$\begin{vmatrix} -2-r & 1 & -2 \\ 1 & -2-r & 2 \\ 3 & -3 & 5-r \end{vmatrix} = 0$$

$$r^3 - r^2 + 5r - 3 = 0 \Rightarrow r_1 = 3; \quad r_{2,3} = -1$$

$$r_1 = 3: \begin{pmatrix} -5 & 1 & -2 \\ 1 & -5 & 2 \\ 3 & -3 & 2 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \cdot e^{3t}$$

$$r_{2,3}: \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 3 & -3 & 6 \end{pmatrix} \quad \begin{matrix} h_1 = 1 \\ \mu_1 = 2 \\ \sigma_1 = 2 \end{matrix}$$

κ_{11}	κ_{12}

$$(\mathbf{A} - r\mathbf{E}) \cdot \kappa_{11} = 0 \Rightarrow \kappa_{11} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad (\mathbf{A} - r\mathbf{E}) \cdot \kappa_{12} = 0 \Rightarrow \kappa_{12} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\xi_2 = \kappa_{11} \cdot e^{-t}; \quad \xi_3 = \kappa_{12} \cdot e^{-t}$$

$$\begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} & 0 \\ -e^{3t} & e^{-t} & 2e^{-t} \\ -3e^{3t} & 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{matrix} x = c_1 e^{3t} + c_2 e^{-t} \\ y = -c_1 e^{3t} + c_2 e^{-t} + 2c_3 e^{-t} \\ z = -3c_1 e^{3t} + c_3 e^{-t} \end{matrix}$$

Príklad č.71:

$$\begin{aligned} x' &= x-3y+3z \\ y' &= -2x-6y+13z \\ z' &= x-4y+8z \end{aligned} \qquad \begin{vmatrix} 1-r & -3 & 3 \\ -2 & -6-r & 13 \\ 1 & -4 & 8-r \end{vmatrix} = 0 \Leftrightarrow r^3-3r^2+3r+1=0 \Leftrightarrow \underline{r_{1,2,3}=1}$$

$$(\mathbf{A}-r\mathbf{E}) = \begin{pmatrix} 0 & -3 & 3 \\ -2 & -7 & 13 \\ 1 & -4 & 7 \end{pmatrix} \quad \begin{aligned} h_1 &= 2 \\ \mu_1 &= 1 \\ \sigma_1 &= 1 \end{aligned}$$

κ_{31}
κ_{21}
κ_{11}

$$(\mathbf{A}-r\mathbf{E})^2 = \begin{pmatrix} 3 & 9 & -18 \\ 1 & -3 & -6 \\ 1 & 3 & -6 \end{pmatrix} \quad \begin{aligned} h_2 &= 1 \\ \mu_2 &= 2 \\ \sigma_2 &= 1 \end{aligned}$$

$$(\mathbf{A}-r\mathbf{E})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \begin{aligned} h_3 &= 0 \\ \mu_3 &= 3 \\ \sigma_3 &= 1 \end{aligned}$$

$$(\mathbf{A}-r\mathbf{E})^3 \cdot \kappa_{31} = 0 \Rightarrow \kappa_{31} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\mathbf{A}-r\mathbf{E}) \cdot \kappa_{31} = \kappa_{21} \Rightarrow \kappa_{21} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

$$(\mathbf{A}-r\mathbf{E})^2 \cdot \kappa_{21} = \kappa_{11} \Rightarrow \kappa_{11} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot e^t$$

$$\xi_2 = \left[\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot t + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right] \cdot e^t \; ; \quad \xi_3 = \left[\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{t^2}{2} + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} \cdot t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot e^t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e^t & 3te^t & \left(\frac{3}{2}t^2+1\right)e^t \\ e^t & (t-2)e^t & \left(\frac{1}{2}t^2-2t\right)e^t \\ e^t & (t-1)e^t & \left(\frac{1}{2}t^2-t\right)e^t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$x = 3c_1e^t + 3tc_2e^t + \left(\frac{3}{2}t^2+1\right)c_3e^t$$

$$y = c_1e^t + (t-2)c_2e^t + \left(\frac{1}{2}t^2-2t\right)c_3e^t$$

$$z = c_1e^t + (t-1)c_2e^t + \left(\frac{1}{2}t^2-t\right)c_3e^t$$

Metóda variácie konštánt

Platí vzťah: $\xi(t) = \Phi\mathbf{C} + \Phi \cdot \int \Phi^{-1}(t) \cdot \overrightarrow{\beta(t)} \cdot dt$

Príklad č.72:

$$\begin{aligned} x' &= -x - 2y + 2e^{-t} \\ y' &= 3x + 4y + e^{-t} \end{aligned} \qquad \vec{\beta} = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$$

$$(\mathbf{A} - \lambda\mathbf{E}) = \begin{pmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{pmatrix} \Rightarrow \det(\mathbf{A} - \lambda\mathbf{E}) = \lambda^2 - 3\lambda + 2$$

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow r_1 = 1;\, r_2 = 2$$

$$r_1 = 1 \Rightarrow (\mathbf{A} - 1\mathbf{E}) = \begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot e^{t}$$

$$r_2 = 2 \Rightarrow (\mathbf{A} - 2\mathbf{E}) = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix} \Rightarrow \xi_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot e^{2t}$$

$$\Phi = \begin{bmatrix} e^{t} & 2e^{2t} \\ -e^{t} & -3e^{2t} \end{bmatrix} \Rightarrow \det\Phi = -3e^{3t} \Rightarrow \Phi^{T} = \begin{bmatrix} e^{t} & -e^{t} \\ 2e^{2t} & -3e^{2t} \end{bmatrix}$$

$$\Phi^{-1} = \frac{1}{\det\Phi} \begin{bmatrix} -3e^{2t} & -2e^{2t} \\ e^{t} & e^{t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} & 2e^{-t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}$$

$$\Phi^{-1} \cdot \vec{\beta}(t) = \begin{pmatrix} 3e^{-t} & 2e^{-t} \\ -e^{-2t} & -e^{-2t} \end{pmatrix} \cdot \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 8e^{-2t} \\ -3e^{-3t} \end{pmatrix}$$

$$\int \Phi^{-1} \cdot \vec{\beta}(t) \cdot dt = \int \begin{pmatrix} 8e^{-2} \\ -3e^{-3t} \end{pmatrix} \cdot dt = \begin{pmatrix} 8\frac{1}{-2}e^{-2} \\ -3\frac{1}{-3}e^{-3t} \end{pmatrix} = \begin{pmatrix} -4e^{-2t} \\ e^{-3t} \end{pmatrix}$$

$$\xi_1(t) = \begin{pmatrix} e^{t} & 2e^{2t} \\ -e^{t} & -3e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -2e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x &= c_1 e^{t} + 2c_2 e^{2t} - 2e^{-t} \\ y &= -c_1 e^{t} - 3c_2 e^{2t} + e^{-t} \end{aligned}$$